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# Mirror duality and noncommutative tori

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## Abstract

In this paper, we study a mirror duality on a generalized complex torus and a noncommutative complex torus. First, we derive a symplectic version of Riemann conditions using mirror duality on ordinary complex tori. Based on this, we will find a mirror correspondence on generalized complex tori and generalize the mirror duality on complex tori to the case of noncommutative complex tori.

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## 1. Introduction

In this paper, we study mirror dualities on complex tori, generalized complex tori and noncommutative complex tori.

Based on string theory, it was proposed in [33] that mirror pairs of Calabi–Yau manifolds admit special Lagrangian torus fibrations over the same base such that generic fibers are dual tori. The mirror duality over elliptic curves is well established in [29], and Kontsevich’s homological mirror symmetry conjecture was first proved in that paper. This is generalized to the Abelian varieties, and in the case of Abelian varieties there is a precise definition of mirror duality which agrees with the suggestion of [33] via dual torus fibrations, see [9, 27, 34]. Also, mirror symmetry on symplectic and complex tori or on Abelian varieties has been studied in many papers such as in [11, 24, 25]. Based on the construction given in [9, 27], we explicitly analyze the relations between Lagrangian submanifolds and the holomorphic line bundles. From this, we derive a symplectic version of Riemann conditions and also find the mirror relation between a complexified symplectic form and a period matrix for the dual lattice. Furthermore, we find a condition for submanifolds of a symplectic torus, which corresponds to a non-holomorphic line bundle on the mirror dual complex torus.

The notion of generalized complex geometry, which was introduced by Hitchin in [14, 15], contains as special cases both complex and symplectic manifolds. It has been studied in [14] that in topological strings on a Calabi–Yau manifold, A-branes, B-branes and other

corresponding notions discussed in [19] are well explained in terms of generalized complex submanifolds. In [18], Kapustin and Orlov studied the notion of an  $N = 2$  superconformal vertex algebra and found a criterion for two different complex tori to produce isomorphic  $N = 2$  superconformal vertex algebras which correspond to mirror duality. In [17], it was discussed that the geometry of topological D-branes is best described using generalized complex structures. In particular, the role of a B-field and the relation between T-duality and an  $N = 2$  superconformal structure are well explained in [17]. Here, we note that a rigorous mathematical formulation of the same in a formal setting was established in [2]. Based on the result in [17] and the analysis made on Abelian varieties, we generalize the mirror duality on complex and symplectic tori to the generalized complex tori. We shall consider a special type of generalized complex torus and define a mirror map between generalized complex tori. Using this mirror map, we verify the mirror correspondence on Abelian varieties. Also, we will discuss the case when a given B-field is not of type (1,1). One may find some related discussions in [1, 4, 12].

The noncommutative tori is known to be the most accessible examples of noncommutative geometry developed by Connes [5]. It also provides the best example in applications of noncommutative geometry to string/M theory, which was initiated in [6]. Analogously, the geometry and gauge theories of a noncommutative torus have been explicitly studied in many papers, such as [6, 7, 30, 31]. A complex geometry of noncommutative torus was developed by Schwarz in [32], and it can be considered as a noncommutative generalization of Abelian varieties. It also provided a basic step to the study of Kontsevich’s homological mirror conjecture [23]. In [16, 26, 28], it has been shown that the conjecture is true for the case of two-dimensional noncommutative tori. Based on the D-brane physics given in [16], we discussed the mathematical aspects of the T-duality on a noncommutative complex torus in [20]. We generalize the mirror correspondence to the higher dimensional cases. Also, we discuss the noncommutative version of Riemann conditions.

## 2. Mirror duality for Abelian varieties

In this section, we briefly review the mirror symmetry on Abelian varieties following [9], [27]. We shall find a necessary and sufficient condition that the mirror dual torus of a symplectic torus becomes an Abelian variety.

Let  $\mathbb{T}^d = \mathbb{C}^g / (\mathbb{Z} \oplus i\mathbb{Z})^g$ ,  $d = 2g$ , be a complex torus equipped with a complexified symplectic 2-form  $\Omega = \omega + i\xi$ . Let  $V \cong \mathbb{R}^d$  be the universal cover of  $\mathbb{T}^d$  and let  $\Gamma = \pi_1(\mathbb{T}^d) \cong \mathbb{Z}^d$ . Also, we assume that there is an  $\Omega$ -Lagrangian linear subspace  $L$  of  $V$  such that  $L \cap \Gamma \cong \mathbb{Z}^g$ , and we can take  $L = i\mathbb{R}^g \subset \mathbb{C}^g$ . Let us consider the Lagrangian torus fibration  $p : V/\Gamma \rightarrow V/(L + \Gamma)$  which admits a section and, hence, we have an isotropic decomposition  $V = V/L \oplus L$  such that  $\Gamma = (\Gamma/\Gamma \cap L) \oplus (\Gamma \cap L)$ . Let  $e_1, \dots, e_g, e_{g+1}, \dots, e_d$  be a basis for the real vector space  $V = V/L \oplus L$ , respectively, such that  $\Omega(e_i, e_j) = \Omega(e_{g+i}, e_{g+j}) = 0$  and  $\Omega(e_{g+i}, e_j) = \mathcal{Z}_{ij}$  for some  $g \times g$  complex matrix  $\mathcal{Z} = (\mathcal{Z}_{ij})$ . By the definition of  $\Omega$ , we may write  $\mathcal{Z} = \text{Re } \mathcal{Z} + i\text{Im } \mathcal{Z}$ , where  $\omega(e_{g+i}, e_j) = (\text{Re } \mathcal{Z})_{ij}$  and  $\xi(e_{g+i}, e_j) = (\text{Im } \mathcal{Z})_{ij}$ . The symplectic form  $\omega$  is positive so that it is nondegenerate. Note that the matrix  $\mathcal{Z}$  can be understood as a linear map from  $V/L$  to  $L^*$ . In other words, we define, using the same notation,  $\mathcal{Z} : V/L \rightarrow L^*$  by  $\mathcal{Z}(v) = \Omega(\cdot, v)$ ,  $v \in V/L$ . Similarly, we have linear maps  $\text{Re } \mathcal{Z}$  and  $\text{Im } \mathcal{Z}$  such that  $\text{Re } \mathcal{Z}(v) = \omega(\cdot, v)$  and  $\text{Im } \mathcal{Z}(v) = \xi(\cdot, v)$  for  $v \in V/L$ . Since the matrix  $\text{Re } \mathcal{Z}$  is nondegenerate, the linear map  $\text{Re } \mathcal{Z} : V/L \rightarrow L^*$  is an isomorphism.

Following the lines of [27] and [9], we construct the mirror dual of  $(\mathbb{T}^d, \Omega)$ . Note that the natural map

$$\alpha : V \oplus V^* \longrightarrow \text{Hom}_{\mathbb{R}}(V, \mathbb{C})$$

defined by  $\alpha(v, v^*)(x) = \Omega(x, v) + iv^*(x)$ ,  $x \in V$ , is an isomorphism of real vector spaces, where  $V^* = \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$  is the dual vector space of  $V$ . There exists a unique complex structure on  $V \oplus V^*$  induced by the isomorphism  $\alpha$ . Let  $L^\perp = \{v^* \in V^* | v^*(l) = 0 \text{ for all } l \in L\}$ . Then  $\alpha$  maps the subspace  $L \oplus L^\perp \subset V \oplus V^*$  to the subspace  $\text{Hom}_{\mathbb{R}}(V/L, \mathbb{C}) \subset \text{Hom}_{\mathbb{R}}(V, \mathbb{C})$ . Passing to the quotient spaces, we get an isomorphism

$$\alpha_L : V/L \oplus L^* \longrightarrow \text{Hom}_{\mathbb{R}}(L, \mathbb{C}) = L^* \otimes_{\mathbb{R}} \mathbb{C},$$

where  $L^* = \text{Hom}_{\mathbb{R}}(L, \mathbb{R})$ . Indeed,  $\alpha_L$  is given as follows:

$$\alpha_L(v + L, l^*)(x) = \Omega(x, v) + il^*(x) = \Omega(x, v) + i\omega(x, y),$$

where we have used the isomorphism  $\omega : V/L \longrightarrow L^*$  defined by  $\omega(y) = \omega(\cdot, y)$ . Let us put

$$(\Gamma \cap L)^\perp = \{\mu \in L^* | \mu(\gamma) \in \mathbb{Z} \text{ for all } \gamma \in \Gamma \cap L\}.$$

Then the mirror of  $(\mathbb{T}^d, \Omega)$  is defined to be

$$(\mathbb{T}^d, \Omega)^\vee = W/\Lambda,$$

where

$$W = (V/L) \oplus L^*, \quad \Lambda = (\Gamma/\Gamma \cap L) \oplus (\Gamma \cap L)^\perp.$$

A complex structure on  $(\mathbb{T}^d, \Omega)^\vee$  is defined uniquely by the isomorphism  $\alpha_L$ . More explicitly, we define a complex structure  $\widehat{J}_\Omega$  on  $V/L \oplus V/L$  which makes the following diagram commute:

$$\begin{array}{ccc} V/L \oplus V/L & \xrightarrow{\alpha_L} & \text{Hom}_{\mathbb{R}}(L, \mathbb{C}) = L^* \otimes_{\mathbb{R}} \mathbb{C} \\ \downarrow j_\Omega & & \downarrow i \\ V/L \oplus V/L & \xrightarrow{\alpha_L} & \text{Hom}_{\mathbb{R}}(L, \mathbb{C}) = L^* \otimes_{\mathbb{R}} \mathbb{C} \end{array}$$

Hence,

$$\begin{aligned} \widehat{J}_\Omega &= \begin{pmatrix} \text{Re } \mathcal{Z} & 0 \\ \text{Im } \mathcal{Z} & \text{Re } \mathcal{Z} \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \text{Re } \mathcal{Z} & 0 \\ \text{Im } \mathcal{Z} & \text{Re } \mathcal{Z} \end{pmatrix} \\ &= \begin{pmatrix} (\text{Re } \mathcal{Z})^{-1} & 0 \\ -(\text{Re } \mathcal{Z})^{-1} \text{Im } \mathcal{Z} (\text{Re } \mathcal{Z})^{-1} & (\text{Re } \mathcal{Z})^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \text{Re } \mathcal{Z} & 0 \\ \text{Im } \mathcal{Z} & \text{Re } \mathcal{Z} \end{pmatrix} \\ &= \begin{pmatrix} -(\text{Re } \mathcal{Z})^{-1} \text{Im } \mathcal{Z} & -1 \\ 1 + (\text{Re } \mathcal{Z})^{-1} \text{Im } \mathcal{Z} (\text{Re } \mathcal{Z})^{-1} \text{Im } \mathcal{Z} & (\text{Re } \mathcal{Z})^{-1} \text{Im } \mathcal{Z} \end{pmatrix}. \end{aligned} \tag{1}$$

For simplicity, let  $A = (\text{Re } \mathcal{Z})^{-1} \text{Im } \mathcal{Z}$ . Then we have

$$\widehat{J}_\Omega = \begin{pmatrix} -A & -1 \\ 1 + A^2 & A \end{pmatrix}. \tag{2}$$

In fact, as a linear map,  $A : V/L \longrightarrow V/L$  is defined by the condition

$$\omega(\cdot, Av) = \xi(\cdot, v), v \in V/L.$$

Let  $f : V/L \longrightarrow L$  be an  $\mathbb{R}$ -linear isomorphism such that  $f(\Gamma/\Gamma \cap L) = \Gamma \cap L$  and let  $L_f = \{f(v) + v | v \in V/L\}$  be the graph of  $f$ , which is a linear subspace of  $V$ . Since  $f(\Gamma/\Gamma \cap L) = \Gamma \cap L$ , the image  $\overline{L}_f$  of  $L_f$  under the projection  $V \longrightarrow V/\Gamma$  intersects

each fiber of  $p : V/\Gamma \rightarrow V/(L + \Gamma)$  in finitely many points. For the basis  $e_1, \dots, e_d$  of  $V = V/L \oplus L$ , let  $f(e_j) = \sum_{k=1}^g f_{kj} e_{g+k}$ . Since

$$\begin{aligned} \Omega(f(e_j) + e_j, f(e_i) + e_i) &= \Omega(f(e_j), e_i) + \Omega(e_j, f(e_i)) \\ &= \Omega\left(\sum_k f_{kj} e_{g+k}, e_i\right) - \Omega\left(\sum_k f_{ki} e_{g+k}, e_j\right) \\ &= \sum_k \{f_{kj} Z_{ki} - f_{ki} Z_{kj}\} \\ &= (Z^t f)_{ij} - (Z^t f)_{ji}, \end{aligned}$$

the graph  $L_f$  of  $f$  is an  $\Omega$ -Lagrangian subspace of  $V$  if and only if  $Z^t f : V/L \rightarrow (V/L)^*$  is symmetric. Analogously, we have the following relations:

$$(\text{Re } Z)^t f = f^t (\text{Re } Z), \quad (\text{Im } Z)^t f = f^t (\text{Im } Z). \tag{3}$$

Associated with the linear map  $f : V/L \rightarrow L$ , we define an antisymmetric bilinear form  $E_f$  on  $(\Gamma/\Gamma \cap L) \oplus (\Gamma \cap L)^\perp$  as follows. For  $u_1, u_2 \in \Gamma/\Gamma \cap L$  and  $v_1^*, v_2^* \in (\Gamma \cap L)^\perp$ ,

$$E_f((u_1, v_1^*), (u_2, v_2^*)) = v_2^*(f(u_1)) - v_1^*(f(u_2)).$$

We extend  $E_f$  to an  $\mathbb{R}$ -bilinear anti-symmetric form on  $V/L \oplus L^*$ . Using the identification  $\text{Re } Z : V/L \cong L^*$ , we consider the bilinear form  $E_f$  as the one on  $V/L \oplus V/L$ . Then the bilinear form can be represented by

$$E_f = \begin{pmatrix} 0 & f^t (\text{Re } Z) \\ -(\text{Re } Z)^t f & 0 \end{pmatrix}.$$

We now show that the graph  $L_f$  of  $f$  is an  $\Omega$ -Lagrangian subspace of  $V$  if and only if the bilinear form  $E_f$  satisfies  $E_f(\widehat{J}_\Omega v, \widehat{J}_\Omega w) = E_f(v, w)$ , for  $v, w \in V/L$ . Suppose that  $L_f$  is  $\Omega$ -Lagrangian, then relations (3) hold. By the definition of  $\widehat{J}_\Omega$  given in (2), we have

$$\begin{pmatrix} -A & -1 \\ 1 + A^2 & A \end{pmatrix}^t \begin{pmatrix} 0 & f^t (\text{Re } Z) \\ -(\text{Re } Z)^t f & 0 \end{pmatrix} \begin{pmatrix} -A & -1 \\ 1 + A^2 & A \end{pmatrix} := \begin{pmatrix} X & Y \\ -Y^t & W \end{pmatrix},$$

where

$$X = -A^t f^t (\text{Re } Z)(1 + A^2) + (1 + A^{2t})(\text{Re } Z)^t f A \tag{4}$$

$$Y = -A^t f^t (\text{Re } Z)A + (1 + A^{2t})(\text{Re } Z)^t f \tag{5}$$

$$W = -f^t (\text{Re } Z)A + A^t (\text{Re } Z)^t f. \tag{6}$$

Since  $A = (\text{Re } Z)^{-1} \text{Im } Z$  and by (3), we have

$$W = -f^t (\text{Re } Z)A + A^t (\text{Re } Z)^t f = -f^t \text{Im } Z + (\text{Im } Z)^t f = 0. \tag{7}$$

By (3) and (7), we have

$$\begin{aligned} X &= -A^t f^t (\text{Re } Z)(1 + A^2) + (1 + A^{2t})(\text{Re } Z)^t f A \\ &= -A^t (\text{Re } Z)^t f (1 + A^2) + (1 + A^{2t}) f^t (\text{Re } Z) A \\ &= -A^t (\text{Re } Z)^t f A^2 + A^{2t} f^t (\text{Re } Z) A \\ &= -A^t ((\text{Re } Z)^t f A + A^t f^t (\text{Re } Z)) A = 0. \end{aligned}$$

Using (7) again, we have

$$\begin{aligned} Y &= -A^t f^t (\text{Re } Z)A + (1 + A^{2t})(\text{Re } Z)^t f \\ &= -A^t f^t (\text{Re } Z)A + A^{2t} (\text{Re } Z)^t f + (\text{Re } Z)^t f \\ &= -A^t f^t (\text{Re } Z)A + A^t f^t (\text{Re } Z)A + (\text{Re } Z)^t f \\ &= (\text{Re } Z)^t f. \end{aligned}$$

Thus, by (3),

$$\begin{pmatrix} X & Y \\ -Y^t & W \end{pmatrix} = \begin{pmatrix} 0 & f^t(\text{Re } \mathcal{Z}) \\ -(\text{Re } \mathcal{Z})^t f & 0 \end{pmatrix} \tag{8}$$

as desired. Conversely, suppose  $E_f(\widehat{J}_\Omega v, \widehat{J}_\Omega w) = E_f(v, w)$ , then (8) is true, and we show that relations (3) are also true. Since  $W = 0$  and by (7), we have  $f^t \text{Im } \mathcal{Z} = \text{Im } \mathcal{Z}^t f$ . Also,  $f^t \text{Im } \mathcal{Z} = \text{Im } \mathcal{Z}^t f$  implies that  $Y = (\text{Re } \mathcal{Z})^t f$ . Thus, by relation (8), we should have  $f^t(\text{Re } \mathcal{Z}) = (\text{Re } \mathcal{Z})^t f$ . Now, the graph  $L_f$  is an  $\Omega$ -Lagrangian subspace of  $V$ .

Finally, we shall show how to find an  $\Omega$ -Lagrangian submanifold of  $\mathbb{T}^d$  from a holomorphic line bundle over  $(\widehat{\mathbb{T}}^d, \Omega)$ . This will allow us to compare the  $\Omega$ -Lagrangian property given in (3) with the Riemann conditions (cf [13]). Also, as we will see in the following section, the argument given here is easily applied to the case of the noncommutative tori.

A holomorphic line bundle on  $\widehat{\mathbb{T}}^d$  is specified by its first Chern class which can be represented by an anti-symmetric bilinear form on  $\Lambda$ . Let  $E_f$  be any integral anti-symmetric bilinear form on  $W = V/L \oplus L^*$ . Without loss of generality, we may assume that  $E_f$  is given by the matrix  $\begin{pmatrix} 0 & f^t \\ -f & 0 \end{pmatrix}$  for the basis given above. Note that we also regard  $f$  as a linear map from  $V/L$  to  $L$ . Associated with such a matrix  $E_f$ , one can find a complex  $g \times g$  matrix  $Z$  such that the  $g \times d$  matrix  $(Z \ f^t)$  is a period matrix over the lattice  $\Lambda$ . Now, the 2-form  $E_f$  on  $\widehat{\mathbb{T}}^d$  is of type (1, 1) if and only if

$$(Z \ f^t) \begin{pmatrix} 0 & -f^{-1} \\ f^{-t} & 0 \end{pmatrix} \begin{pmatrix} Z^t \\ f \end{pmatrix} = 0,$$

which implies that  $Z = Z^t$ . From the period matrix, we can reconstruct the complex structure  $\widehat{J}_\Omega$  in the same basis. Note that the complex structure given in (2) is defined on  $V/L \oplus V/L$  using the identification  $\text{Re } \mathcal{Z} : V/L \cong L^*$ . Thus, we must consider the period matrix using the basis for  $V/L \oplus V/L$  and this is done using  $\text{Re } \mathcal{Z}$ . Now, in order to find the complex structure  $\widehat{J}_\Omega$ , we need to solve the matrix system

$$(Z \ f^t)\widehat{J}_\Omega = i(Z \ f^t)$$

or equivalently

$$\begin{pmatrix} \text{Re } Z & f^t \text{Re } \mathcal{Z} \\ \text{Im } Z & 0 \end{pmatrix} \widehat{J}_\Omega = \begin{pmatrix} -\text{Im } Z & 0 \\ \text{Re } Z & f^t \text{Re } \mathcal{Z} \end{pmatrix}.$$

For simplicity, we let  $\mathfrak{f} = f^t \cdot \text{Re } \mathcal{Z}$ . If we consider the matrix  $f$  is regarded as the linear map  $f : V/L \rightarrow L$ , the  $\mathfrak{f}$  is regarded as the linear map from  $V/L$  to  $(V/L)^*$ , using the identification  $\text{Re } \mathcal{Z} : V/L \cong L^*$ . Now, we have

$$\begin{aligned} \widehat{J}_\Omega &= \begin{pmatrix} \text{Re } Z & f^t \text{Re } \mathcal{Z} \\ \text{Im } Z & 0 \end{pmatrix}^{-1} \begin{pmatrix} -\text{Im } Z & 0 \\ \text{Re } Z & f^t \cdot \text{Re } \mathcal{Z} \end{pmatrix} \\ &= \begin{pmatrix} 0 & (\text{Im } Z)^{-1} \\ \mathfrak{f}^{-1} & -\mathfrak{f}^{-1} \text{Re } Z (\text{Im } Z)^{-1} \end{pmatrix} \begin{pmatrix} -\text{Im } Z & 0 \\ \text{Re } Z & f^t \cdot \text{Re } \mathcal{Z} \end{pmatrix} \\ &= \begin{pmatrix} (\text{Im } Z)^{-1} \text{Re } Z & (\text{Im } Z)^{-1} \mathfrak{f} \\ -\mathfrak{f}^{-1} [\text{Im } Z + \text{Re } Z (\text{Im } Z)^{-1} \text{Re } Z] & -\mathfrak{f}^{-1} \text{Re } Z (\text{Im } Z)^{-1} \mathfrak{f} \end{pmatrix} \tag{9} \end{aligned}$$

Formula (9) is also given in [25]. By comparing (9) and (2), we get the following consistency relations:

$$A = -(\text{Im } Z)^{-1} \text{Re } Z = -\mathfrak{f}^{-1} \text{Re } Z (\text{Im } Z)^{-1} \mathfrak{f} \tag{10}$$

$$1 = -(\text{Im } Z)^{-1} \mathfrak{f} \tag{11}$$

$$1 + A^2 = -f^{-1}[\text{Im } Z + \text{Re } Z(\text{Im } Z)^{-1}\text{Re } Z]. \tag{12}$$

It is easy to check that the conditions (11) and (12) are equivalent. By (11), we have

$$f = -\text{Im } Z \tag{13}$$

and also by (10) and (13)

$$fA = -f(\text{Im } Z)^{-1}\text{Re } Z = -\text{Re } Z(\text{Im } Z)^{-1}f = \text{Re } Z.$$

Thus, we see that if  $Z = Z^t$ , then  $f$  and  $fA$  are symmetric. Similarly, if  $Z = \overline{Z}^t$ , then  $f$  is skew-symmetric and  $fA$  is symmetric.

Conversely, suppose that  $f$  and  $fA$  are symmetric. Then by (13),  $\text{Im } Z$  is symmetric. From (10) and (11), we have  $fA = -f(\text{Im } Z)^{-1}\text{Re } Z = \text{Re } Z$ . Thus, if  $fA$  is symmetric, then  $\text{Re } Z$  is symmetric, which implies that  $Z = Z^t$ . Also, we see that  $\text{Im } Z > 0$  if and only if  $f < 0$ , by (11). Thus, we see that the Riemann conditions on the complex side are the mirror dual to the  $\Omega$ -Lagrangian property given in (3). On the other hand, if  $f$  is skew-symmetric and  $fA$  is symmetric, then we have  $Z = \overline{Z}^t$ . Thus, in this case, we see that a non-holomorphic bundle on the mirror dual torus corresponds to a submanifold of the original torus, which is defined to be the graph of a skew-symmetric linear map after the identification with  $\text{Re } \mathcal{Z} : V/L \cong L^*$ .

In particular, an interesting fact is that the correspondence between complexified symplectic form on  $\mathbb{T}^d$  and the complex structure on the mirror dual torus is easily seen by the relations

$$f = f^t \text{Re } \mathcal{Z} = -\text{Im } Z, \quad fA := f^t \text{Im } \mathcal{Z} = \text{Re } Z \tag{14}$$

and, hence,

$$\begin{aligned} f^t \mathcal{Z} &= f^t \cdot \text{Re } \mathcal{Z} + i f^t \cdot \text{Im } \mathcal{Z} \\ &= -\text{Im } Z + i \text{Re } Z \\ &= i(\text{Re } Z + i \text{Im } Z) = iZ. \end{aligned}$$

As a conclusion, the mirror dual complex torus  $(\widehat{\mathbb{T}}^d, \widehat{J}_\Omega)$ , equipped with the integral 2-form  $E_f$ , is an Abelian variety if and only if the real matrices  $f, fA$  are symmetric and  $f < 0$ . This might be understood as a symplectic version of the Riemann condition. Also, for a holomorphic line bundle  $\widehat{L}$  on  $\widehat{\mathbb{T}}^d$  such that  $c_1(\widehat{L}) \in H^{1,1}(\widehat{\mathbb{T}}^d, \mathbb{R}) \cap H^2(\widehat{\mathbb{T}}^d, \mathbb{Z})$ , we may write  $c_1(\widehat{L})$  as an integral bilinear form  $E_f = \begin{pmatrix} 0 & f^t \\ -f & 0 \end{pmatrix}$  on  $W = V/L \oplus L^*$ . Then the graph of the integral linear map  $f : V/L \rightarrow L$  is an  $\Omega$ -Lagrangian subspace of  $V$ . This analysis will be generalized to the case of noncommutative complex tori in the following section.

### 3. Mirror duality on generalized complex tori

The aim of this section is to rephrase the mirror duality given in section 2 in terms of generalized complex structures which were introduced by Hitchin ([15] and see also [14]). Modifying the notion ‘T-duality in all direction’ defined by Kapustin in [17], we define T-duality in half direction, and we will show that the duality is well matched with the mirror symmetry given in section 2.

Let us first recall the definition of a generalized complex structure on a real vector space. Let  $V$  be a  $d$ -dimensional vector space over  $\mathbb{R}$ . Then the space  $V \oplus V^*$  is naturally equipped with a pseudo-Euclidean metric defined by

$$\langle X + v^*, Y + w^* \rangle = \frac{1}{2}(v^*(Y) + w^*(X)) = \frac{1}{2}(Xv^*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} Y \\ w^* \end{pmatrix}, \tag{15}$$

for  $X, Y \in V$  and  $v^*, w^* \in V^*$ . For simplicity, we write  $q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  as the pseudo-Euclidean metric given in (15). A *generalized complex structure* on  $V$  is an endomorphism  $\mathcal{J}$  of  $V \oplus V^*$  satisfying  $\mathcal{J}^2 = -1$  and  $\mathcal{J}\mathcal{J}^t = -1$ , i.e.,  $\mathcal{J}$  is orthogonal with respect to the pseudo-Euclidean metric. This orthogonality can be seen as the following commuting diagram:

$$\begin{array}{ccc} V^* \oplus V & \xrightarrow{\mathcal{J}^t} & V^* \oplus V \\ \downarrow q & & \downarrow q \\ V \oplus V^* & \xrightarrow{-\mathcal{J}} & V \oplus V^* \end{array}$$

A generalized complex torus  $(\mathbb{T}^d, \mathcal{J}_1, \mathcal{J}_2)$  is a real torus  $\mathbb{T}^d = V/\Gamma$  equipped with a pair  $(\mathcal{J}_1, \mathcal{J}_2)$  of generalized complex structures on the direct sum of the tangent and cotangent bundles making it a fiberwise generalized complex vector space. In particular, such a pair of generalized complex structures is called a generalized Kähler structure on  $\mathbb{T}^d$ . A typical example is given as follows. Let  $J \in \text{End}(V)$  be a complex structure on  $\mathbb{T}^d = V/\Gamma$  endowed with a constant Kähler form  $\omega$  with a B-field  $\xi$ , and with a flat Riemannian metric  $g$ . Then, we have two generalized complex structures:

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix}, \quad \mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix},$$

and we may transform them by a B-field  $\xi$ :

$$\begin{aligned} \mathcal{J}_J^\xi &= \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix} \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\xi & 1 \end{pmatrix} = \begin{pmatrix} J & 0 \\ \xi J + J^t \xi & -J^t \end{pmatrix} \\ \mathcal{J}_\omega^\xi &= \begin{pmatrix} 1 & 0 \\ \xi & 1 \end{pmatrix} \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\xi & 1 \end{pmatrix} = \begin{pmatrix} \omega^{-1} \xi & -\omega^{-1} \\ \omega + \xi \omega^{-1} \xi & -\xi \omega^{-1} \end{pmatrix}. \end{aligned}$$

It is easy to check that  $(\mathbb{T}^d, \mathcal{J}_J^\xi, \mathcal{J}_\omega^\xi)$  is a generalized complex torus. Now, such two generalized complex tori  $(\mathbb{T}_1^d, \mathcal{J}_{J_1}^{\xi_1}, \mathcal{J}_{\omega_1}^{\xi_1})$  and  $(\mathbb{T}_2^d, \mathcal{J}_{J_2}^{\xi_2}, \mathcal{J}_{\omega_2}^{\xi_2})$  are mirror of each other if there is a lattice isomorphism  $\phi : \Gamma_1 \oplus \Gamma_1^* \rightarrow \Gamma_2 \oplus \Gamma_2^*$  such that  $\phi^t q_2 \phi = q_1$  and  $\phi^{-1} \mathcal{J}_{J_1}^{\xi_1} \phi = \mathcal{J}_{J_2}^{\xi_2}, \phi^{-1} \mathcal{J}_{\omega_1}^{\xi_1} \phi = \mathcal{J}_{\omega_2}^{\xi_2}$ , where  $q_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, i = 1, 2$ , are the pseudo-Euclidean metric.

We now rephrase the mirror duality given in section 2 in terms of generalized Kähler structure by constructing an explicit mirror map  $\phi$  and the map will be referred as a T-duality in half direction since it corresponds to a relative T-duality along the fibers of a special Lagrangian fibration, (compare with [17]). Let  $\mathbb{T}^{2g} = V/\Gamma = \mathbb{C}^g/(\mathbb{Z} \oplus i\mathbb{Z})$  be a complex torus equipped with a complexified symplectic form  $\Omega = \omega + i\xi$ . Then the mirror of  $(\mathbb{T}^{2g} = V/\Gamma, \Omega)$  is given by  $W/\Lambda$ , where  $W = (V/L) \oplus L^*$  and  $\Lambda = (\Gamma/\Gamma \cap L) \oplus (\Gamma \cap L)^\perp$  by decomposing  $\Gamma = (\Gamma/\Gamma \cap L) \oplus (\Gamma \cap L)$ . We define

$$\phi = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} : \Gamma \oplus \Gamma^* \rightarrow \Lambda \oplus \Lambda^*. \tag{16}$$

We shall verify the map  $\phi$  gives the mirror correspondence discussed in section 2. Note that we have a generalized Kähler structure on  $\mathbb{T}^{2g} = V/\Gamma$  is given by

$$\mathcal{J}_J^\xi = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{17}$$



$$\mathcal{J}_\omega^\xi = \begin{pmatrix} \omega^{-1}\xi & -\omega^{-1} \\ \omega + \xi\omega^{-1}\xi & -\xi\omega^{-1} \end{pmatrix}. \tag{18}$$

Equation (17) follows from the fact that  $\xi$  is a type of (1, 1) with respect to the canonical complex structure  $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . Using the notation given in section 2, the entries in (18) are given as follows:

$$\begin{aligned} \omega^{-1}\xi &= \begin{pmatrix} (\text{Re } \mathcal{Z})^{-1}\text{Im } \mathcal{Z} & 0 \\ 0 & (\text{Re } \mathcal{Z})^{-t}\text{Im } \mathcal{Z}' \end{pmatrix} \\ -\omega^{-1} &= \begin{pmatrix} 0 & -(\text{Re } \mathcal{Z})^{-1} \\ (\text{Re } \mathcal{Z})^{-t} & 0 \end{pmatrix} \\ \omega + \xi\omega^{-1}\xi &= \begin{pmatrix} 0 & -\text{Re } \mathcal{Z}' - \text{Im } \mathcal{Z}'(\text{Re } \mathcal{Z})^{-t}\text{Im } \mathcal{Z}' \\ \text{Re } \mathcal{Z} + \text{Im } \mathcal{Z}(\text{Re } \mathcal{Z})^{-1}\text{Im } \mathcal{Z} & 0 \end{pmatrix} \\ -\xi\omega^{-1} &= \begin{pmatrix} -\text{Im } \mathcal{Z}'(\text{Re } \mathcal{Z})^{-t} & 0 \\ 0 & -\text{Im } \mathcal{Z}(\text{Re } \mathcal{Z})^{-1} \end{pmatrix}. \end{aligned}$$

Then it is easy to compute

$$\phi^{-1}\mathcal{J}_J^\xi\phi = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\phi^{-1}\mathcal{J}_\omega^\xi\phi = -\begin{pmatrix} \widehat{J}_\Omega & 0 \\ 0 & -\widehat{J}_\Omega^t \end{pmatrix},$$

where  $\widehat{J}_\Omega$  is the complex structure on the mirror dual torus given in section 2.

Using the mirror map defined above, we may consider the case that the complex 2-form  $\Omega = \omega + i\xi$  is not of type (1,1). Since we have  $\xi$  as a B-field, we shall keep  $\omega$  as of type (1,1). With the same basis  $e_1, \dots, e_g, e_{g+1}, \dots, e_d$  for the vector space  $V$  given in section 2, we define a complexified symplectic form  $\Omega$  as follows:  $\Omega(e_i, e_j) = \sqrt{-1}X_{ij}$ ,  $\Omega(e_{g+i}, e_j) = \mathcal{Z}_{ij}$ ,  $\Omega(e_{g+i}, e_{g+j}) = 0$ , where  $X_{ij}$  is real. Then we may represent  $\xi$  as a block matrix  $\xi = \begin{pmatrix} X & -\text{Im } \mathcal{Z}' \\ \text{Im } \mathcal{Z} & 0 \end{pmatrix}$ . In other words, we only have common  $\omega$  and  $\xi$ -Lagrangian subspaces on the base space of the Lagrangian torus fibrations considered in section 2. One finds that  $\xi$  is no more of type (1, 1). To be more precise, we have a canonical complex structure  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and consider the following general form:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} X & -\text{Im } \mathcal{Z}' \\ -\text{Im } \mathcal{Z} & Y \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} Y & -\text{Im } \mathcal{Z} \\ \text{Im } \mathcal{Z}' & X \end{pmatrix}.$$

From the relation above, we see that the 2-form, which is represented by the matrix  $\begin{pmatrix} X & -\text{Im } \mathcal{Z}' \\ -\text{Im } \mathcal{Z} & Y \end{pmatrix}$  is of type (1, 1) if and only if  $X = Y$  and the matrix  $\text{Im } \mathcal{Z}$  is symmetric. Also,  $\xi$  is of type (2, 0) or (0, 2) if and only if  $X = -Y$  and  $\text{Im } \mathcal{Z}$  is anti-symmetric. Then since

$$\xi = \begin{pmatrix} X & -\text{Im } \mathcal{Z}' \\ \text{Im } \mathcal{Z} & 0 \end{pmatrix} = \begin{pmatrix} \frac{X}{2} & \frac{\text{Im } \mathcal{Z} - \text{Im } \mathcal{Z}'}{2} \\ \frac{\text{Im } \mathcal{Z} - \text{Im } \mathcal{Z}'}{2} & -\frac{X}{2} \end{pmatrix} + \begin{pmatrix} \frac{X}{2} & -\frac{\text{Im } \mathcal{Z} + \text{Im } \mathcal{Z}'}{2} \\ \frac{\text{Im } \mathcal{Z} + \text{Im } \mathcal{Z}'}{2} & \frac{X}{2} \end{pmatrix},$$

we see that the  $\xi$  is a most general type of B-field. Now the generalized Kähler structure is given by

$$\mathcal{J}_J^\xi = \begin{pmatrix} J & 0 \\ \xi J + J^t \xi & -J^t \end{pmatrix} \tag{19}$$

$$\mathcal{J}_\omega^\xi = \begin{pmatrix} \omega^{-1}\xi & -\omega^{-1} \\ \omega + \xi\omega^{-1}\xi & -\xi\omega^{-1} \end{pmatrix} \tag{20}$$

and

$$\begin{aligned} \omega^{-1}\xi &= \begin{pmatrix} (\text{Re } \mathcal{Z})^{-1}\text{Im } \mathcal{Z} & 0 \\ -(\text{Re } \mathcal{Z})^{-t}X & (\text{Re } \mathcal{Z})^{-t}\text{Im } \mathcal{Z}^t \end{pmatrix} - \omega^{-1} = \begin{pmatrix} 0 & -(\text{Re } \mathcal{Z})^{-1} \\ (\text{Re } \mathcal{Z})^{-t} & 0 \end{pmatrix} \\ \omega + \xi\omega^{-1}\xi &= \begin{pmatrix} (\text{Im } \mathcal{Z})^t(\text{Re } \mathcal{Z})^{-t}X + X(\text{Re } \mathcal{Z})^{-1}\text{Im } \mathcal{Z} & -\text{Re } \mathcal{Z}^t - \text{Im } \mathcal{Z}^t(\text{Re } \mathcal{Z})^{-t}\text{Im } \mathcal{Z}^t \\ \text{Re } \mathcal{Z} + \text{Im } \mathcal{Z}(\text{Re } \mathcal{Z})^{-1}\text{Im } \mathcal{Z} & 0 \end{pmatrix} \\ -\xi\omega^{-1} &= \begin{pmatrix} -\text{Im } \mathcal{Z}^t(\text{Re } \mathcal{Z})^{-t} & -X(\text{Re } \mathcal{Z})^{-1} \\ 0 & -\text{Im } \mathcal{Z}(\text{Re } \mathcal{Z})^{-1} \end{pmatrix}. \end{aligned}$$

For the complex structure (19), since

$$\begin{aligned} \xi J + J^t \xi &= \begin{pmatrix} X & -\text{Im } \mathcal{Z}^t \\ \text{Im } \mathcal{Z} & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} X & -\text{Im } \mathcal{Z}^t \\ \text{Im } \mathcal{Z} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \text{Im } \mathcal{Z} - \text{Im } \mathcal{Z}^t & -X \\ -X & \text{Im } \mathcal{Z}^t - \text{Im } \mathcal{Z} \end{pmatrix}, \end{aligned}$$

we have

$$\mathcal{J}_J^\xi = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \text{Im } \mathcal{Z} - \text{Im } \mathcal{Z}^t & -X & 0 & -1 \\ -X & \text{Im } \mathcal{Z}^t - \text{Im } \mathcal{Z} & 1 & 0 \end{pmatrix}.$$

By applying the mirror map, we get

$$\phi^{-1} \mathcal{J}_J^\xi \phi = \begin{pmatrix} 0 & 0 & 0 & 1 \\ X & 0 & -1 & \text{Im } \mathcal{Z}^t - \text{Im } \mathcal{Z} \\ \text{Im } \mathcal{Z} - \text{Im } \mathcal{Z}^t & 1 & 0 & X \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Similarly, we have

$$\phi^{-1} \mathcal{J}_\omega^\xi \phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$\begin{aligned} A &= \begin{pmatrix} (\text{Re } \mathcal{Z})^{-1}\text{Im } \mathcal{Z} & (\text{Re } \mathcal{Z})^{-1} \\ -\text{Re } \mathcal{Z} - \text{Im } \mathcal{Z}(\text{Re } \mathcal{Z})^{-1}\text{Im } \mathcal{Z} & -\text{Im } \mathcal{Z}(\text{Re } \mathcal{Z})^{-1} \end{pmatrix} \\ B &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ C &= \begin{pmatrix} (\text{Im } \mathcal{Z})^t(\text{Re } \mathcal{Z})^{-t}X + X(\text{Re } \mathcal{Z})^{-1}\text{Im } \mathcal{Z} & X(\text{Re } \mathcal{Z})^{-1} \\ (\text{Re } \mathcal{Z})^tX & 0 \end{pmatrix} \\ D &= \begin{pmatrix} -\text{Im } \mathcal{Z}^t(\text{Re } \mathcal{Z})^{-t} & \text{Re } \mathcal{Z}^t + \text{Im } \mathcal{Z}^t(\text{Re } \mathcal{Z})^{-t}\text{Im } \mathcal{Z}^t \\ -(\text{Re } \mathcal{Z})^{-t} & (\text{Re } \mathcal{Z})^{-t}\text{Im } \mathcal{Z}^t \end{pmatrix}. \end{aligned}$$

Finally, we note that the B-field  $\xi$  is mapped to  $\phi\xi\phi^{-1}$  under the mirror map  $\phi$  and we get the above correspondence.

Note that the generalized Kähler structure  $\mathcal{J}_J^\xi$  given in (19) is block-lower triangular. Thus, if we take ‘T-duality in all directions’ as defined in [17], the corresponding generalized Kähler structure on the dual torus becomes a block-upper triangular. However, this is impossible unless the B-field  $\xi$  is of type (1,1) related to an  $N = 2$  superconformal field theory. Hence

if the (0,2)-part of  $\xi$  is not 0, then the T-duality has a problem. Analogously, since T-duality between a complex torus and its dual torus makes the categories of B-branes on both tori equivalent, the category of B-branes should be twisted by the (0,2)-part of  $\xi$ . From this, Kapustin proposed that such a twistedness is characterized by the fact that the matrix  $\mathcal{J}_J^\xi$  is block-lower-triangular, see [17] for details.

We have chosen a specific type of B-field which has a nonzero (0,2)-part, and it defines a twistedness on a B-brane category on the complex torus as discussed above. On the other hand, from our choice of the B-field  $\xi$ , we find that the A-brane category on  $\mathbb{T}^{2g}$  should be deformed by the (0,2)-part of  $\xi$ , since the generalized Kähler structure  $\mathcal{J}_\omega^\xi$  given in (20) is block-lower-triangular after taking a T-duality in half directions as shown in the above computations. As a conclusion, we state a symplectic version of Kapustin’s proposal: noncommutative deformations are characterized by the fact that the mirror dual of a symplectic type of the generalized Kähler structure such as the one given in (20), is block-lower-triangular, and the A-brane category is twisted by the (0,2)-part of the given B-field.

#### 4. Mirror duality on noncommutative complex tori

In this section, we generalize the mirror duality on Abelian varieties to the case of noncommutative complex tori. Let us first recall some basic facts for a noncommutative complex torus and holomorphic structures on it, see [32] for details. A noncommutative torus  $\widehat{\mathbb{T}}_\theta^d$  is generated by  $d$ -unitaries  $U_1, \dots, U_d$

$$U_i U_j = \exp(2\pi i \theta_{ij}) U_j U_i, \tag{21}$$

where  $\theta = (\theta_{ij})$  is an irrational  $d \times d$  skew-symmetric matrix. The relation (21) defines the presentation of the involutive algebra

$$A_\theta^d = \left\{ \sum_{(n_1, \dots, n_d) \in \mathbb{Z}^d} a_{n_1, \dots, n_d} U_1^{n_1} \cdots U_d^{n_d} \mid a_{n_1, \dots, n_d} \in \mathcal{S}(\mathbb{Z}^d) \right\},$$

where the coefficient function  $(n_1, \dots, n_d) \mapsto a_{n_1, \dots, n_d}$  rapidly decays at infinity. By definition, the algebra  $A_\theta^d$  is the algebra of smooth functions on  $\widehat{\mathbb{T}}_\theta^d$ . The ordinary torus  $\widehat{\mathbb{T}}^d$  acts on the algebra  $A_\theta^d$  (cf [30]), and the infinitesimal form of the action of  $\widehat{\mathbb{T}}^d$  on  $A_\theta^d$  defines a Lie algebra homomorphism

$$\delta : W \longrightarrow \text{Der}(A_\theta^d), \tag{22}$$

where  $W$  is a real vector space which is the Lie algebra of  $\widehat{\mathbb{T}}^d = W/\Lambda$ , for some lattice  $\Lambda$ , and  $\text{Der}(A_\theta^d)$  denotes the Lie algebra of derivations of  $A_\theta^d$ . Generators  $\delta_1, \dots, \delta_d$  of  $\text{Der}(A_\theta^d)$  act as follows:

$$\delta_j(U_j) = 2\pi i U_j \quad \text{and} \quad \delta_i(U_j) = 0 \quad \text{for} \quad i \neq j.$$

A noncommutative torus  $\widehat{\mathbb{T}}_\theta^d$  is said to be a *noncommutative complex torus* if the Lie algebra  $W \cong \mathbb{R}^d$  is equipped with a complex structure. Associated with a given complex structure on  $W$ , the complexification  $W \otimes_{\mathbb{R}} \mathbb{C}$  can be decomposed by two complex conjugate subspaces  $W^{0,1}$  and  $W^{1,0}$ , which are of a complex dimension  $g$  such that  $2g = d$ . Let

$$\Omega^{0,p} = \wedge^p (W^{0,1})^* \quad \text{and} \quad \Omega^{0,\bullet} = \bigoplus_{p=0}^g \Omega^{0,p}.$$

A holomorphic structure on a vector bundle  $\mathcal{E}$  over  $\widehat{\mathbb{T}}_\theta^d$ , which corresponds to a finitely generated projective (left)  $A_\theta$ -module, is given by a linear map

$$\bar{\nabla} : \mathcal{E} \otimes \Omega^{0,\bullet} \longrightarrow \mathcal{E} \otimes \Omega^{0,\bullet+1}$$

such that, for  $\alpha, \beta \in W^{0,1}$ ,

$$\bar{\nabla}_\alpha(u \cdot e) = u \cdot \bar{\nabla}_\alpha e + \bar{\delta}_\alpha(u) \cdot e, \quad u \in A_\theta, \quad e \in \mathcal{E} \tag{23}$$

and

$$[\bar{\nabla}_\alpha, \bar{\nabla}_\beta] = 0, \tag{24}$$

where  $\bar{\delta}_1, \dots, \bar{\delta}_g$  are generators for the Lie algebra  $\text{Der}(A_\theta^d)$  associated with a basis for  $W^{0,1}$ . From the condition (24), we get a complex

$$0 \longrightarrow \mathcal{E} \xrightarrow{\bar{\nabla}} \mathcal{E} \otimes (W^{0,1})^* \xrightarrow{\bar{\nabla}} \mathcal{E} \otimes \wedge^2(W^{0,1})^* \longrightarrow \dots,$$

and the corresponding cohomology will be denoted by  $H^*(\mathcal{E}, \bar{\nabla})$ . Note that, since  $\dim_{\mathbb{C}} W^{0,1} = g$ ,  $H^k(\mathcal{E}, \bar{\nabla}) = 0$  if  $k > g$ . In particular,  $H^0(\mathcal{E}, \bar{\nabla})$  consists of  $\phi \in \mathcal{E}$  such that  $\bar{\nabla}\phi = 0$ . The elements of  $H^0(\mathcal{E}, \bar{\nabla})$  are called *holomorphic vectors* or *theta vectors*.

The vector bundles over  $\widehat{\mathbb{T}}^d$  are classified by the  $K$ -theory of  $\widehat{\mathbb{T}}^d$ , and there is a ring homomorphism  $\text{ch} : K^*(\widehat{\mathbb{T}}^d) \longrightarrow H^*(\widehat{\mathbb{T}}^d, \mathbb{Q})$ . Similarly, finitely generated projective  $A_\theta^d$ -modules are classified by  $K_0(A_\theta^d)$ , and the Chern character takes values in  $H^*(\widehat{\mathbb{T}}^d, \mathbb{R})$ . The targets of the both Chern characters are related to the deformation parameter  $\theta \in \wedge^2 W$ . The relation is summarized by the following diagram:

$$\begin{array}{ccc} K^0(\widehat{\mathbb{T}}^d) & \xrightarrow{\text{ch}} & H^{\text{even}}(\widehat{\mathbb{T}}^d, \mathbb{Q}) \\ & & \downarrow e^{i(\theta)} \\ K_0(A_\theta^d) & \xrightarrow{\text{Ch}} & H^{\text{even}}(\widehat{\mathbb{T}}^d, \mathbb{R}) \end{array}$$

where  $i(\theta)$  denotes the contraction with 2-vector  $\theta$  and  $e^{i(\theta)}$  is defined using the usual power series with the exterior product as done in [8]. More precisely, for a finitely generated projective  $A_\theta^d$ -module  $\mathcal{E}$ , the Chern character is defined to be

$$\text{Ch}(\mathcal{E}) = \sum_{k=0} \frac{1}{k!} \widehat{\text{tr}}(F^k) \cdot \frac{1}{(2\pi i)^k}, \tag{25}$$

where  $F$  is the curvature of a connection  $\nabla$  on  $\mathcal{E}$  and  $\widehat{\text{tr}}$  is the trace on the algebra  $\text{End}_{A_\theta^d}(\mathcal{E})$ . Then by [theorem 4.5] [30], one can construct a finitely generated projective  $A_\theta^d$ -module  $\mathcal{E}$  and a connection  $\nabla$  on it such that

$$\text{Ch}(\mathcal{E}) = e^{i(\theta)} \text{ch}(E). \tag{26}$$

Note that the cohomology group  $H^\bullet(\widehat{\mathbb{T}}^d, \mathbb{R})$  can be identified with the exterior algebra  $\wedge^\bullet W^*$ , where  $W^* = \text{Hom}_{\mathbb{R}}(W, \mathbb{R})$  is the dual vector space of  $W$ . In below, we shall study the mirror dual property of the cohomological deformation described above.

Let  $f : V/L \longrightarrow L$  be an integral linear map and let  $L_f = \{f(v) + v | v \in V/L\}$  be the graph of  $f$ . Since  $f$  is integral,  $L_f \cap \Gamma \cong \mathbb{Z}^g$  and  $\bar{L}_f = L_f / (L_f \cap \Gamma)$  intersects each fiber of  $p : V/\Gamma \longrightarrow V/(L + \Gamma)$  at one point. As we have discussed in section 2, the linear subspace  $L_f$  defines an integral antisymmetric bilinear form  $E_f = \begin{pmatrix} 0 & f^i \\ -f & 0 \end{pmatrix}$  on  $W = V/L \oplus L^*$ . Then the linear subspace  $L_f$  of  $V$  is an  $\Omega$ -Lagrangian, if and only if  $E_f$  can be seen as an element in  $H^{1,1}(\widehat{\mathbb{T}}^d, \mathbb{C}) \cap H^2(\widehat{\mathbb{T}}^d, \mathbb{Z})$ . Then there is a holomorphic line bundle  $\widehat{L}_f$  on  $\widehat{\mathbb{T}}^d$  such that

$c_1(\widehat{L}_f) = E_f$ . Let us denote the intersection number of Lagrangians  $\overline{L}_f$  and  $\overline{L}$  in  $V/\Gamma$  by  $\#(\overline{L}_f \cap \overline{L})$ . Since  $\#(\overline{L}_f \cap \overline{L}) = 1$ , it is easy to see that

$$\text{Pf } E_f = \#(\overline{L}_f \cap \overline{L}),$$

where  $\text{Pf } E_f$  is the Pfaffian of the anti-symmetric form  $E_f$  and  $\overline{L}$  is the image of  $V/L$  under the covering map  $V \rightarrow V/\Gamma$ . Note that the moduli space of the flat Lagrangian submanifolds of  $\mathbb{T}^d$  parallel to  $L_f/L_f \cap \Gamma$  is identified with the  $d$ -dimensional torus.

Based on the construction given in [30], we shall deform the line bundle  $\widehat{L}_m$  on  $\widehat{\mathbb{T}}^d$  to a holomorphic bundle over the noncommutative torus  $\widehat{\mathbb{T}}_\theta^d$ . A finitely generated projective  $A_\theta^d$ -module, which is in fact a bundle over  $\widehat{\mathbb{T}}_\theta^d$ , is given by a Schwarz space  $\mathcal{S}(\mathbb{R}^g \times G)$ , where  $G$  is a finite Abelian group. Let  $G = \prod_{i=1}^g \mathbb{Z}_{m_i}$  and let  $\mathcal{E} = \mathcal{S}(\mathbb{R}^g \times G)$ , where  $m_1 m_2 \cdots m_g = \text{Pf } E_f$  corresponds to the degree of the line bundle  $\widehat{L}_f$ . Using the representation of the Heisenberg commutation relations for the finite group  $G$ , one can find unitary operators  $W_i$  acting on  $\mathcal{S}(G) = \mathbb{C}^{m_1} \otimes \cdots \otimes \mathbb{C}^{m_g}$  such that

$$W_i W_j = \exp[2\pi i (E_f^{-1})_{ij}] W_j W_i. \tag{27}$$

The operators  $W_j$  can also be obtained using the twist eating solution studied in [10], and such operators specify the line bundle  $\widehat{L}_f$ . In order to define an  $A_\theta^d$ -module action on  $\mathcal{E} = \mathcal{S}(\mathbb{R}^g \times G)$ , one needs to consider an embedding of the lattice  $\Lambda$  into  $\mathbb{R}^g \times (\mathbb{R}^g)^*$  in the sense of [30]. Such an embedding map can be given by a real invertible  $d \times d$  matrix  $T$ , satisfying the relation

$$T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} T^t = \gamma, \tag{28}$$

where  $\gamma$  is an irrational skew-symmetric matrix such that  $E_f^{-1} - \gamma = \theta$ . Let us denote the first and the second  $g$  rows in the  $j$ th column of the matrix  $T$  by  $T_{[1,j]}$  and  $T_{[2,j]}$ , respectively. Then associated with the embedding  $T$ , we define operators  $V_j$  on  $\mathcal{S}(\mathbb{R}^g)$  by

$$(V_j h)(\mathbf{s}) = \exp(2\pi i \mathbf{s}^t \cdot T_{[2,j]}) h(\mathbf{s} + T_{[1,j]}^t), \tag{29}$$

where  $\mathbf{s} = (s_1 \cdots s_g)^t \in \mathbb{R}^g$  and  $h \in \mathcal{S}(\mathbb{R}^g)$ . Then the operators satisfy the following commutation relation:

$$V_i V_j = \exp(-2\pi i \gamma_{ij}) V_j V_i. \tag{30}$$

Combining (27) and (30), the unitary operators  $U_i = V_i \otimes W_i$  define an  $A_\theta^d$ -module action on  $\mathcal{E}$ . Thus, we get a vector bundle  $\mathcal{E}$  on the noncommutative torus  $\widehat{\mathbb{T}}_\theta^d$ . Similarly, one can construct a constant curvature connection  $\nabla$  on  $\mathcal{E}$  using the inverse matrix  $T^{-1}$ . More explicitly, define

$$(\nabla_j h)(\mathbf{s}) = 2\pi i \mathbf{s}^t \cdot T_{[1,j]}^{-1} h(\mathbf{s}) - \frac{\partial h}{\partial \mathbf{s}} \cdot T_{[2,j]}^{-1}, \tag{31}$$

where  $\frac{\partial h}{\partial \mathbf{s}} = \left( \frac{\partial h}{\partial s_1} \cdots \frac{\partial h}{\partial s_g} \right)$ . Then we have

$$[\nabla_i, \nabla_j] = 2\pi i (\gamma^{-1})_{ij}. \tag{32}$$

Note that the operators of the form  $\nabla_i + R_i$ ,  $R_i \in \mathbb{R}$  also satisfy the commutation relation (32), and thus the moduli space of such connections is a  $d$ -dimensional torus.

Let us consider the case when the curvature  $\gamma^{-1}$  is given by the following simple block matrix:

$$\gamma^{-1} = \begin{pmatrix} 0 & F_{\gamma^{-1}}^t \\ -F_{\gamma^{-1}} & 0 \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} 0 & -F_\gamma \\ \mathcal{F}_\gamma^t & 0 \end{pmatrix}, \tag{33}$$

where  $F_\gamma$  is a  $g \times g$  real matrix such that at least one of its entries is irrational and  $F_{\gamma^{-1}} = F_\gamma^{-1}$ . Note that the matrix  $F_\gamma$  depends on the choice of a basis for  $W$ . Associated with the curvature  $\gamma^{-1}$  as given above, we define an antisymmetric bilinear form  $E_{\gamma^{-1}}$  on  $W = V/L \oplus L^*$  as follows. For  $v_1, v_2 \in V/L$  and  $l_1^*, l_2^* \in L^*$ ,

$$E_{\gamma^{-1}}((v_1, l_1^*), (v_2, l_2^*)) = l_2^*(f_{\gamma^{-1}}(v_1)) - l_1^*(f_{\gamma^{-1}}(v_2)), \tag{34}$$

where  $f_{\gamma^{-1}} : V/L \rightarrow L$  is a linear map whose matrix is given by  $F_{\gamma^{-1}}$  with respect to the given basis for  $W$ . Thus, the curvature  $\gamma^{-1}$  is the corresponding matrix for the antisymmetric form  $E_{\gamma^{-1}}$  on  $W$ . Also, we may regard the antisymmetric form  $E_{\gamma^{-1}}$  on  $W$  as an element of the cohomology group  $H^2(\widehat{\mathbb{T}}^d, \mathbb{R})$  in the following way. Since  $\text{Ch}(\mathcal{E}) = e^{i(\theta)} \text{ch}(\widehat{L}_f) \in H^{2\bullet}(\widehat{\mathbb{T}}^d, \mathbb{R})$ , we may write

$$\text{Ch}(\mathcal{E}) = \text{Ch}_0(\mathcal{E}) + \text{Ch}_1(\mathcal{E}) + \dots \in H^0(\widehat{\mathbb{T}}^d, \mathbb{R}) \oplus H^2(\widehat{\mathbb{T}}^d, \mathbb{R}) \oplus \dots$$

The curvature  $E_{\gamma^{-1}}$  of  $\nabla$  is given by  $2\pi i \text{Ch}_1(\mathcal{E})/\text{Ch}_0(\mathcal{E})$ , and the 0th cohomology is computed to be

$$\text{Ch}_0(\mathcal{E}) = \dim(\mathcal{E}) = \det T = \text{Pf}(\gamma).$$

The first cohomology class is now given by  $\text{Ch}_1(\mathcal{E}) = \dim(\mathcal{E})E_{\gamma^{-1}} \in H^2(\widehat{\mathbb{T}}^d, \mathbb{R}) = \wedge^2 W^*$ . Thus, the curvature  $E_{\gamma^{-1}}$ , which is normalized by the dimension of  $\mathcal{E}$ , can be understood as an anti-symmetric form on  $W = V/L \oplus L^*$  representing  $\text{Ch}_1(\mathcal{E})$ .

Recall that the vector space  $W$  is equipped with the complex structure  $\widehat{J}_\Omega$ , which is defined in section 2. The linear map  $\widehat{J}_\Omega$  is defined on  $V/L \oplus V/L$  and by identifying  $V/L$  with  $L^*$  via  $\text{Re } \mathcal{Z}$ , it defines a complex structure on  $W$ . Now by definition,  $(\widehat{\mathbb{T}}_\theta^d, \widehat{J}_\Omega)$  is a noncommutative complex torus. We shall define a holomorphic structure on the vector bundle  $\mathcal{E}$  on  $\widehat{\mathbb{T}}_\theta^d$ , which is compatible with the complex structure  $\widehat{J}_\Omega$ . Let  $W \otimes_{\mathbb{R}} \mathbb{C} = W^{1,0} \oplus W^{0,1}$ , where  $W^{1,0}$  and  $W^{0,1}$  are  $i$  and  $-i$  eigenspaces of  $\widehat{J}_\Omega$ , respectively. Associated with the integral anti-symmetric bilinear form  $E_f$ , which is the first Chern class of the line bundle  $\widehat{L}_f$ , one has a period matrix of  $\Lambda \subset W$ , as discussed in section 2. Along with the deformation of the Chern character of  $\widehat{L}_f$ , we also deform the period matrix. More explicitly, since  $\text{Ch}_1(\mathcal{E}) \in \wedge^2 W^*$ , for the given basis for  $W$ , we may write

$$\text{Ch}_1(\mathcal{E}) = \frac{1}{2} \sum_{i < j} \text{Pf}(\gamma) (F_{\gamma^{-1}}^t)_{ij} dx_i \wedge dx_j,$$

where  $x_1, \dots, x_d$  are the dual coordinates on  $W$ . Then the period matrix of  $\Lambda \subset W$  is deformed to  $\mathcal{U}_\gamma = (F_\gamma^t \mathbf{Z})$ , where  $\mathbf{Z}$  is a complex  $g \times g$  matrix. By this period-like matrix, we make the change of the basis for  $W$  into the one for  $W^{0,1}$ . Thus, we let

$$\bar{\delta} = \mathcal{U}_\gamma \delta \quad \text{or} \quad \bar{\delta}_a = \sum_{j=1}^d (\mathcal{U}_\gamma)_{aj} \delta_j.$$

According to the basis change, we define a holomorphic connection  $\bar{\nabla}$  on  $\mathcal{E}$  by  $\bar{\nabla} = \mathcal{U}_\gamma \nabla$ . Then

$$\begin{aligned} [\bar{\nabla}, \bar{\nabla}] &= [\mathcal{U}_\gamma \nabla, \mathcal{U}_\gamma \nabla] \\ &= \mathcal{U}_\gamma \gamma^{-1} \mathcal{U}_\gamma^t \\ &= (F_\gamma^t \mathbf{Z}) \begin{pmatrix} 0 & F_{\gamma^{-1}}^t \\ -F_{\gamma^{-1}} & 0 \end{pmatrix} \begin{pmatrix} F_\gamma \\ \mathbf{Z}^t \end{pmatrix} \\ &= -\mathbf{Z} + \mathbf{Z}^t. \end{aligned}$$

Thus,  $[\bar{\nabla}, \bar{\nabla}] = 0$  if and only if  $Z = Z^t$ . In other words, the connection  $\bar{\nabla}$  defines a holomorphic structure on  $\mathcal{E}$  if and only if  $Z$  is symmetric. We note that a holomorphic structure on  $\mathcal{E}$  corresponds to a real 2-form which is of type (1,1), as we will see in below. Also, we may define a structure which corresponds to a real 2-form, which is of type (2,0) or (0,2). Using the holomorphic connection  $\bar{\nabla}$ , we define a structure of type (2,0) or (0,2) as  $[\bar{\nabla}, \nabla] = 0$ , and it is easy to see that  $Z = \bar{Z}^t$  if  $[\bar{\nabla}, \nabla] = 0$ .

Associated with the holomorphic structure  $\bar{\nabla}$  on  $\mathcal{E}$ , we may recover the compatible complex structure  $\widehat{J}_\Omega$  on  $\widehat{\mathbb{T}}_\theta^d$ , which is inherited from that of  $\widehat{\mathbb{T}}^d$ . First, note that the derivations are a noncommutative generalization of derivatives  $\frac{\partial}{\partial x}$ . On the other hand, the complex structure  $\widehat{J}_\Omega$  in section 2 is represented by using differential forms. Thus, in order to get a matrix form for  $\widehat{J}_\Omega$  on  $\widehat{\mathbb{T}}_\theta^d$ , we need to take the dual period-like matrix  $\bar{U}^*$ , and the relation between  $\bar{U}$  and  $\bar{U}^*$  is given as follows:

$$\begin{pmatrix} \bar{U}^* \\ \bar{U} \end{pmatrix} = \begin{pmatrix} F_\gamma^t & Z \\ F_\gamma^t & \bar{Z} \end{pmatrix}^{-t} = \frac{1}{2i} \begin{pmatrix} -\text{Im} Z^{-t} & 0 \\ 0 & \text{Im} Z^{-t} \end{pmatrix} \begin{pmatrix} \bar{Z}^t & F_\gamma \\ Z^t & F_\gamma \end{pmatrix} \begin{pmatrix} F_{\gamma^{-1}} & 0 \\ 0 & -F_{\gamma^{-1}} \end{pmatrix}.$$

Thus, by a simple change of basis, we may set  $\bar{U}^* = (ZF_\gamma)$  and as in section 2, we solve the matrix equation

$$(ZF_\gamma)\widehat{J}_\Omega = i(ZF_\gamma).$$

Using the identification  $\text{Re } \mathcal{Z} : V/L \cong L^*$ , as in section 2, we have

$$(ZF_\gamma \cdot \text{Re } \mathcal{Z})\widehat{J}_\Omega = i(ZF_\gamma \cdot \text{Re } \mathcal{Z})$$

and, hence,

$$\widehat{J}_\Omega = \begin{pmatrix} -(\text{Im } Z)^{-1} \text{Re } Z & -(\text{Im } Z)^{-1} F_\gamma \\ F_\gamma^{-1} [\text{Im } Z + \text{Re } Z (\text{Im } Z)^{-1} \text{Re } Z] & F_\gamma^{-1} \text{Re } Z (\text{Im } Z)^{-1} F_\gamma \end{pmatrix}, \quad (35)$$

where  $F_\gamma = F_\gamma \cdot \text{Re } \mathcal{Z}$ . Also, by relations (2) and (35), we get

$$A = -(\text{Im } Z)^{-1} \text{Re } Z = F_\gamma^{-1} \text{Re } Z (\text{Im } Z)^{-1} F_\gamma^t.$$

Furthermore, since  $Z$  is symmetric, we have the same relations as in (14):

$$F_\gamma \cdot \text{Re } \mathcal{Z} = \text{Im } Z \text{ and } F_\gamma \cdot \text{Im } \mathcal{Z} = \text{Re } Z.$$

Now by the same analysis given in section 2, the matrix  $Z$  is symmetric if and only if the graph of the linear map  $f_{\gamma^{-1}} : V/L \rightarrow L$  is an  $\omega$ -Lagrangian subspace of  $V$ . Equivalently, the graph is a Lagrangian subspace of  $V$  if and only if the connection  $\bar{\nabla}$  defines a compatible connection on  $\mathcal{E}$  on the noncommutative complex torus  $(\widehat{\mathbb{T}}_\theta^d, \widehat{J}_\Omega)$ . Also, we see that the antisymmetric bilinear form  $E_{\gamma^{-1}}$ , defined above, satisfies the relation

$$E_{\gamma^{-1}}(\widehat{J}_\Omega v, \widehat{J}_\Omega w) = E_{\gamma^{-1}}(v, w), \quad v, w \in W. \quad (36)$$

From relation (36), we find that the graph  $L_{\gamma^{-1}} = \{f_{\gamma^{-1}}(v) + v | v \in V/L\}$  of the linear isomorphism  $f_{\gamma^{-1}} : V/L \rightarrow L$  corresponds to the real 2-form  $E_{\gamma^{-1}}$  of type (1,1), and this implies that the linear Lagrangian subspace  $L_{\gamma^{-1}}$  of  $V$  is associated with a holomorphic structure on  $\widehat{\mathbb{T}}_\theta^d$  via mirror duality. However,  $L_\gamma \cap \Gamma$  is not isomorphic to  $\mathbb{Z}^g$ . In other words, the image  $\overline{L_{\gamma^{-1}} + \Gamma}$  of  $L_{\gamma^{-1}}$  under the covering map  $V \rightarrow V/\Gamma$  is not compact and is isomorphic to  $\mathbb{R}^g$  in  $\mathbb{T}^d$ .

Let us consider the case when the curvature  $\gamma^{-1}$  is of the most general form. Since  $\gamma^{-1}$  is skew-symmetric, there is an orthogonal matrix  $O$  such that

$$O\gamma^{-1}O^t = \begin{pmatrix} 0 & \Delta_{\gamma^{-1}} \\ -\Delta_{\gamma^{-1}} & 0 \end{pmatrix},$$

where  $\Delta_{\gamma^{-1}}$  is a real  $g \times g$  diagonal matrix. Let  $\mathcal{U}_O = (\Delta_{\gamma^{-1}}^{-1} \mathbf{Z}) \cdot O$  and let  $\bar{\nabla} = \mathcal{U}_O \nabla$ . Then we have

$$\begin{aligned} [\bar{\nabla}, \bar{\nabla}] &= [\mathcal{U}_O \nabla, \mathcal{U}_O \nabla] \\ &= \mathcal{U}_O \gamma^{-1} \mathcal{U}_O^t \\ &= (\Delta_{\gamma^{-1}}^{-1} \mathbf{Z}) \cdot O \gamma^{-1} O^t \begin{pmatrix} \Delta_{\gamma^{-1}}^{-1} \\ \mathbf{Z}^t \end{pmatrix} \\ &= (\Delta_{\gamma^{-1}}^{-1} \quad \mathbf{Z}) \begin{pmatrix} 0 & \Delta_{\gamma^{-1}} \\ -\Delta_{\gamma^{-1}} & 0 \end{pmatrix} \begin{pmatrix} \Delta_{\gamma^{-1}}^{-1} \\ \mathbf{Z}^t \end{pmatrix}. \end{aligned}$$

Thus,  $\bar{\nabla}$  defines a holomorphic structure on  $\mathcal{E}$  if and only if  $\mathbf{Z} = \mathbf{Z}^t$ . In this case, associated with the matrix  $\begin{pmatrix} 0 & \Delta_{\gamma^{-1}} \\ -\Delta_{\gamma^{-1}} & 0 \end{pmatrix}$ , we have the Lagrangian subspace  $L_\Delta$  of  $V$ , which is defined to be the graph of the linear map  $\Delta_{\gamma^{-1}} : V/L \rightarrow L$ . Thus, the corresponding Lagrangian subspace of  $V$  for  $\gamma^{-1}$  is given by the rotation of  $L_\Delta$  by the orthogonal transformation  $O$ .

Finally, we shall find holomorphic vectors for the vector bundle  $\mathcal{E}$  over  $\widehat{\mathbb{T}}_\theta^d$ . Recall that the holomorphic vectors are elements of  $H^0(\mathcal{E}, \bar{\nabla})$ , the kernel of the linear map  $\bar{\nabla} : \mathcal{E} \rightarrow \mathcal{E} \otimes (W^{0,1})^*$ . Using the Euclidean metric for  $W \cong \mathbb{R}^d$ , we may assume that the curvature matrix  $\gamma^{-1}$  is of the form given in (33). Then a compatible holomorphic structure on  $\mathcal{E}$  is specified by  $\bar{\nabla} = \mathcal{U}_\gamma \nabla$ , where  $\mathcal{U}_\gamma = \begin{pmatrix} F_\gamma^t \\ \mathbf{Z} \end{pmatrix}$  as given above. By the definition of  $\nabla$  given in (31), we have, for  $\phi \in \mathcal{E}$  and  $\mathbf{s} = (s_1 \cdots s_g)^t \in \mathbb{R}^g$ ,

$$\begin{pmatrix} \bar{\nabla}_1 \phi(\mathbf{s}) \\ \vdots \\ \bar{\nabla}_g \phi(\mathbf{s}) \end{pmatrix} = \begin{pmatrix} F_\gamma^t & \mathbf{Z} \end{pmatrix} \begin{pmatrix} \nabla_1 \phi(\mathbf{s}) \\ \vdots \\ \nabla_d \phi(\mathbf{s}) \end{pmatrix} = \begin{pmatrix} F_\gamma^t & \mathbf{Z} \end{pmatrix} T^{-t} \begin{pmatrix} -(\frac{\partial \phi}{\partial \mathbf{s}})^t \\ 2\pi i \mathbf{s} \phi(\mathbf{s}) \end{pmatrix},$$

where  $\frac{\partial \phi}{\partial \mathbf{s}} = (\frac{\partial \phi}{\partial s_1} \cdots \frac{\partial \phi}{\partial s_g})$ . Thus, if  $\bar{\nabla} \phi(\mathbf{s}) = 0$ , then we get a system of the first-order linear partial differential equations:

$$2\pi i (ZT_{[22]}^{-t} + F_\gamma^t T_{[12]}^{-t}) \cdot \mathbf{s} - (ZT_{[21]}^{-t} + F_\gamma^t T_{[11]}^{-t}) \cdot \left(\frac{\partial \phi}{\partial \mathbf{s}}\right)^t = 0, \tag{37}$$

where  $T_{[ij]}^{-t}$  denotes the  $g \times g$  matrix, which constitutes the  $(i, j)$  block of  $T^{-t}$ . Then the solution for system (37) is

$$\phi(\mathbf{s}) = \exp[\pi i \mathbf{s}^t \cdot (ZT_{[21]}^{-t} + F_\gamma^t T_{[11]}^{-t})^{-1} (ZT_{[22]}^{-t} + F_\gamma^t T_{[12]}^{-t}) \cdot \mathbf{s}].$$

Note that the set  $M_T$  of all embeddings  $T$  satisfying relation (28) is the moduli space of finitely generated projective modules over  $\widehat{\mathbb{T}}_\theta^d$ , equipped with a constant curvature connection  $\nabla$  such that  $[\nabla, \nabla] = 2\pi i \gamma^{-1}$ , and such a connection is defined in terms of  $T^{-t}$ . Equivalently, the space  $M_T$  is in one-to-one correspondence with the moduli space of constant curvature connections on  $\mathcal{E}$  whose curvature is  $2\pi i \gamma^{-1}$ . Furthermore, if  $\nabla_0$  is a connection on  $\mathcal{E}$  such that  $[\nabla_0, \nabla_0] = 2\pi i \gamma^{-1}$ , then all other connections satisfying the curvature condition are given in the form  $\nabla = \nabla_0 + \mathbf{r}$ , where  $\mathbf{r} \in \mathbb{R}^d$ . Now, we shall define a connection  $\nabla_0$  using the specific embedding  $T = \begin{pmatrix} F_\gamma^t & 0 \\ 0 & 1 \end{pmatrix}$  or  $T^{-t} = \begin{pmatrix} F_\gamma^{-t} & 0 \\ 0 & 1 \end{pmatrix}$ . Then the corresponding holomorphic structure is given by  $\bar{\nabla}_0 = \mathcal{U}_\gamma \nabla_0$ , and the holomorphic vector is

$$\phi(\mathbf{s}) = \exp[\pi i \mathbf{s}^t \cdot \mathbf{Z} \cdot \mathbf{s}].$$



Note that such a holomorphic vector exists only if the complex matrix  $Z$  is symmetric, and thus if the bundle does not admit a holomorphic structure then the holomorphic vector does not exist. In general, for the connection  $\nabla = \nabla_0 + \mathbf{r}$ , the holomorphic connection is analogously defined and the holomorphic vector is computed to be

$$\phi(\mathbf{s}) = \exp[\pi i \mathbf{s}^t \cdot Z \cdot \mathbf{s} + 2\pi i \mathbf{s} \cdot \mathcal{U}_\gamma \cdot \mathbf{r}].$$

Thus, the solution  $\phi$  is in the Schwarz space  $\mathcal{S}(\mathbb{R}^g)$  only when  $\text{Im } Z > 0$ .

## 5. Summary and prospects

In this paper, we have studied mirror duality on Abelian varieties, which has been well established. By an explicit study of the known results, we could find an exact mirror correspondence between complexified symplectic form and the complex structure for the mirror dual torus. Also, we have reinterpreted the Riemann conditions in terms of Lagrangian submanifolds, and we were able to find a symplectic condition for non-holomorphic bundle over the mirror dual torus. We found in section 4 that all the above-mentioned results are naturally generalized to the case of holomorphic noncommutative complex torus. The result for non-holomorphic bundle shed some lights to the study of non-holomorphic noncommutative complex torus, and we will address this problem later [21]. Associated with this problem, we described the mirror structure for a generalized complex torus with a canonical complex structure in section 2. First, we rephrased the mirror duality on Abelian varieties in terms of generalized complex structures in the case when the given B-field was of type (1,1). Also, we discussed the case when the B-field was of a general type. This might be a first step for us to go for studying Kontsevich's homological mirror symmetry for Abelian varieties, equipped with a general type of the B-field, as was indicated in [17]. We will study this problem too later ([22]) with the categorical approach using the Lie algebroid structure as studied in [3].

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